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Abstract

We consider functions that assign to each evaluation profile a preference system or a list of menu-dependent preferences. The rule by which such an assignment takes place is said to be a menu-dependent preference functional (MDPFL). We extend the concepts of invariance under individual cardinal transformations, weak Pareto, binary independence, weak dictatorship, and veto power from the context of social welfare functional to our framework of MDPFLs. We consider admissible sets of evaluation profiles that are slightly more general than necessarily requiring that all evaluation profiles be admissible. We introduce the concepts of nested and nested* MDPFLs. Our first result says that a nested MDPFL that is invariant under individual transformations, globally weakly Paretian, and satisfies global binary independence must be weakly dictatorial. Our second result says that a nested* MDPFL that is invariant under individual transformations, globally weakly Paretian, and satisfies global binary independence must have an individual/criterion that wields veto power.

Keywords: Menu dependence; Preference functional; Nested; Nested*; Weakly dictatorial; Veto power.

1. INTRODUCTION

As indicated by Rubinstein (2012), "when a rational decision maker forms a preference relation, it is often on the basis of more primitive relations. For example, the choice of a PC may depend on considerations such as 'size of memory', 'ranking by PC magazine', and 'price'. Each of these considerations expresses a preference relation on the set of PCs." If the underlying problem is one associated with aggregation of evaluations according to several criteria into a preference relation over alternatives, then contrary to the approach adopted by Arrow and Raynaud (1986), not only the rankings along each criterion but also the magnitude of the evaluations may play a vital role. For instance, when it comes to evaluating PCs, the trade-offs between more memory and a higher price may depend not only on the fact that one has to pay a higher price for an expanded memory size but also upon the magnitude of the increments. Consider two brands of PCs—brand x and brand y—and suppose that we are interested in two criteria, namely memory and price. Suppose the size of x's memory is larger than the size of y's memory, whereas y is cheaper than x. However, for the same difference in the size of memory, it matters a lot what the difference in the prices of y and x are. If y costs a thousand rupees less than x, then one has a better reason to prefer x over y, than if y costs rupees fifty thousand rupees less than x. This cannot be effectively captured unless we have numerical representation of our preferences along each criterion that expresses the intensity of preference along that criterion. The seminal work of Sen (1970) addresses this issue via a social welfare functional that allows for interpersonal comparison of utility in the context of social choice. Using utility functions instead of mere preference orderings allows us to extract more information (by way of intensity of preference) and thus expands the informational basis of social choice. Our context includes aggregating evaluations over several criteria and is not restricted to aggregating the individual preferences of members of a society into a social preference relation. Why we require this expanded context will be apparent soon.

In the traditional Arrowian framework that Sen (1970) extends, it is assumed that collective preferences are menu-independent. Although this assumption may be meaningful when aggregating individual preferences into a social preference relation, its appropriateness often becomes doubtful when evaluations along several criteria are aggregated into the preferences of a single decision maker. The reason for this is neither the presence of several criteria along which evaluations are considered nor in the process of aggregation. The reason why menu-independence may be violated is because, the preferences of an individual human agent often defies menu independence. According to Sen (1993), the underlying menu of alternatives that is offered for choice can give us information about the underlying situation and this may influence the preferences of the decision maker over the menu of alternatives. The following example due to Luce and Raiffa (1957) illustrates the point. A person after arriving in an unknown city goes to a restaurant to have his dinner. On asking the waiter about what is available for the main course, he is told that he has to choose between broiled salmon and steak because those are the only two items on offer. He decides to have broiled salmon. Soon after taking the order, the waiter returns with the information that (contrary to what he had said earlier) snails and frog legs are also available. On hearing this, the visitor asks for steak instead of broiled salmon. The visitor concluded on the basis of experience that only good restaurants serve snails and frog legs and so it would be safe to eat steak that he really prefers to broiled salmon. Without being sure about the quality of the restaurant and hence of the food it serves, he was being cautious while ordering broiled salmon instead of steak. This type of choice behavior reveals menu-dependent preferences on the part of the decision maker. A list of such menu-dependent preferences that we following Tyson (2008) refer to as a preference system has been discussed in Sen (1997) where the following example appears. "If invited to tea (t) by an acquaintance you might accept the invitation rather than going home (O), that is, pick t from the choice over {t, O}, and yet turn the invitation down if the acquaintance, whom you do not know very well, offers you a wider menu of either having tea with him, or some heroin and cocaine (h), that is, you may pick O, rejecting t, from the larger set {t, h, O}. The expansion of the menu offered by this acquaintance may tell you something about the kind of person he is, and this could affect your decision even to have tea with him."

In this paper, we consider two different types of preference systems: nested and nested*. Nested preference systems trace their origin to the work of Tyson (2008), whereas nested* preference systems have been defined in Cato (2014). A preference system is nested if whenever an alternative x is at least as good as an alternative y in a menu then x is at least as good as alternative y in any larger menu. A preference system is nested* if whenever alternative x is at least as good as alternative y in a menu, then x is at least as good as y in any smaller menu that contains both. As in the existing literature, the star (*) above "nested" in "nested*" is used to distinguish the two crucial properties we use in the paper.

Our approach in this paper consists of assigning to each evaluation profile a preference system. The rule by which such an assignment takes place is said to be a menu-dependent preference functional (MDPFL). We extend the concepts of invariance under individual cardinal transformations, weak Pareto, binary independence, weak dictatorship, and veto power from the context of social welfare functional to our framework of MDPFLs. We consider admissible sets of evaluation profiles that are slightly more general than necessarily requiring that all evaluation profiles be admissible. We say that an MDPFL is nested if it always assigns a nested preference system and we say that a MDPFL is nested* if it always assigns a nested* preference system.

Our first result says that a nested MDPFL that is invariant under individual cardinal transformations, is globally weakly Paretian, and satisfies global binary independence must be weakly dictatorial. Our second result says that a nested* MDPFL that is invariant under individual cardinal transformations, is globally weakly Paretian, and satisfies global binary independence must have an individual/criterion that wields veto power.

An evaluation function is always a numerical representation of a reflexive, complete, and transitive binary relation and any strictly increasing transformation of the evaluation function is a numerical representation of the same reflexive, complete, and transitive binary relation. As the underlying set of alternatives is finite, any reflexive, complete, and transitive binary relation can be numerically represented by an evaluation function and any strictly increasing transformation of the evaluation function is a numerical representation of the same reflexive, complete, and transitive binary relation. Thus, to study the results in the original Arrowian framework (which is precisely what Cato (2014) does) using evaluation functions, we would have to require the MDPFL to satisfy the property of assigning the same preference system to any two evaluation profiles, where the two evaluation functions of the individual/criteria were strictly increasing transformations of each other. The Sen (1970) framework, in which our discussion is located, requires the MDPFL to use more information contained in the evaluation profile than what the Arrowian framework used by Cato (2014) would require. Thus, as in Sen (1970), our results require the strictly weaker assumption that says that the MDPFL satisfies the property of assigning the same preference system to any two evaluation profiles, where the two evaluation functions of the individual/criteria were strictly increasing and *affine* transformations of each other. It is this property that we refer to as invariance under individual cardinal transformations. Thus, apart from a minor generalization of the universal/unrestricted domain condition, our main result is a generalization of the one in Cato (2014) in much the same way that the result in Sen (1970) that we are referring to is a generalization of the impossibility theorem of Arrow.

Finally, we need to justify the title of the paper that carries forward a long-lasting tradition in social choice or group decision theory. Since, nowhere else in the paper is there a reference to impossibility except in the title, what do we have in mind that is/are impossible? The impossibility results are a rephrased version of the two main results of this paper. The first impossibility result says that it is impossible to find a nested MDPFL that is invariant under individual cardinal transformations, is globally weakly Paretian, satisfies global binary independence, and does not allow a dictator. Our second impossibility result says that it is impossible to find a nested* MDPFL that is invariant under individual cardinal transformations, is globally weakly Paretian, satisfies global binary independence, and does not allow an individual/criterion that wields veto power.

2. FRAMEWORK

Our framework is almost the same as the one in Cato (2014). Let X be a nonempty finite set of alternatives containing at least three elements, and let $\Psi(X)$ be the set of all nonempty subsets of X. Given $A\in\Psi(X)$, let $\mathcal{O}_A$ denote the set of all orderings (reflexive, complete, and transitive binary relations) on A. For simplicity, let $\mathcal{O}_X$ be denoted by $\mathcal{O}$.

Given $A\in\Psi(X)$, let R_A denote a typical member of $\mathcal{O}_A$, which is to be interpreted as a *preference relation over the alternatives in* A. Thus, if for $x,y\in A$ we have $(x,y)\in R_A$ then we say that given A, x is *as good as* y. The asymmetric part of R_A is denoted by $P(R_A)$ and if for $x,y\in A$ we have $(x,y)\in P(R_A)$ then we say that given A, x is *strictly preferred to* y. The symmetric part of R_A is denoted by $I(R_A)$ and if for $x,y\in A$ we have $(x,y)\in I(R_A)$ then we say that given A, x is *indifferent to* y.

Given $A,B\in\Psi(X)$ with $B\subset A$, the restriction of $R_A\in\mathcal{O}_A$ denoted $R_A|B$ is the binary relation $R_A\cap(B\times B)$ in $\mathcal{O}_B$.

A *preference system* is a list $<R_A|A\in\Psi(X)>$ such that for all $A\in\Psi(X)$, $R_A\in\mathcal{O}_A$. Let $\mathcal{S}$ denote the set of all collective preference systems.

The following property of a preference system is due to Tyson (2008).

A preference system $<R_A|A\in\Psi(X)>$ is said to be *nested* if for all $A,B\in\Psi(X)$, $A\subset B$ implies $R_A\subset R_B$.

The following property of a preference system is due to Cato (2014).

A preference system $<R_A|A\in(X)>$ is said to be *nested** if for all $A,B\in\Psi(X)$, $A\subset B$ implies $R_B|A\subset R_A$.

The set of individuals/criteria is $N = \{1,2,\ldots,n\}$.

An *evaluation profile* is a function $U: X\times N\rightarrow\mathbb{R}$ where for each $i\in N$, the function $U(.,i):X\rightarrow\mathbb{R}$ is the evaluation of the alternatives in X by individual i or along criterion i.

Let $\mathcal{U}$ denote the set of all evaluation profiles. An *admissible set (of evaluation profiles)* is a nonempty subset $\mathcal{D}$ of $\mathcal{U}$.

A *menu-dependent preference functional* (MDPFL) *on an admissible set* $\mathcal{D}$ is a function $F:\mathcal{D}\rightarrow\mathcal{S}$. For $U\in\mathcal{D}$, $F(U) = <F_A(U)|A\in\Psi(X)>$ where $F_A(U)$ is the preference relation over the alternatives in A that is assigned by F at U.

Let V be an evaluation profile. An evaluation profile U is said to be an *affine transformation* of V, if for all $i\in N$ there exists $a_i\in\mathbb{R}$ and $b_i\in\mathbb{R}_{++}$ (i.e. strictly positive real numbers) such that $U(.,i) = a_i + b_iV(.,i)$.

Note that since the definition of an affine transformation U of V allows for two degrees of freedom for each individual i, given any two real numbers a and b and any two alternatives $x,y\in X$, we can always find $a_i\in\mathbb{R}$ and $b_i\in\mathbb{R}_{++}$ such that $U(x,i) = \alpha$ and $U(y,i) = \beta$, *provided* either $(V(x,i) - V(y,i))\,(\alpha - \beta) > 0$ or $V(x,i) - V(y,i) = 0 = \alpha - \beta$. In fact under such circumstances $b_i = \dfrac{\alpha-\beta}{V(x,i)-V(y,i)} > 0$ if $(V(x,i) - V(y,i))(\alpha - \beta) > 0$ and any $(a_i, b_i)\in\mathbb{R}\times\mathbb{R}_{++}$ if $V(x,i) - V(y,i) = 0 = \alpha - \beta$.

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An MDPFL F on an admissible set $\mathcal{D}$ is said to be *cardinally noncomparable* (CN) (or *invariant under individual cardinal transformations*) if for all U,V∈$\mathcal{D}$ such that U is an affine transformation of V, we have F(U) = F(V).

An MDPFL F on an admissible set $\mathcal{D}$ is said to be *weakly Paretian* (or satisfy *weak Pareto*) (WP) if for all x,y∈X and U∈$\mathcal{D}$ the following is true: [U(x,i) > U(y,i) for all i∈N] implies [(x,y)∈P(F_A(U)) for A∈Ψ(X) satisfying x,y∈A].

An MDPFL F on an admissible set $\mathcal{D}$ is said to be *globally weakly Paretian* (or satisfy *global weak Pareto*) (GWP) if for all x,y∈X and U∈$\mathcal{D}$ the following is true: [U(x,i) > U(y,i) for all i∈N] implies [(x,y)∈P(F_X(U))].

Clearly GWP implies WP

An MDPFL F on an admissible set $\mathcal{D}$ is said to be *binarily invariant* (or satisfy *binary independence*) (BIN) if for all A∈Ψ(X), x,y∈A and U,V∈$\mathcal{D}$ the following is true: [U(.,i)|{x,y} = V(.,i)|{x,y} for all i∈N] implies [F_A(U)|{x,y} = F_A(V)|{x,y}].

In the above [U(.,i)|{x,y} = V(.,i)|{x,y} for all i∈N] means [U(x,i) = V(x,i) and U(y,i) = V(y,i) for all i∈N].

An MDPFL F on an admissible set $\mathcal{D}$ is said to be *globally binarily independent* (or satisfy *global binary independence*) (GBIN) if for all x,y∈X and U,V∈$\mathcal{D}$ the following is true: [U(.,i)|{x,y} = V(.,i)|{x,y} for all i∈N] implies [F_X(U)|{x,y} = F_X(V)|{x,y}].

Clearly, GBIN implies BIN.

An MDPFL F on an admissible set $\mathcal{D}$ is said to be *nested* if for all U∈$\mathcal{D}$, F(U) = <F_A(U)|A∈Ψ(X)> is nested.

An MDPFL F on an admissible set $\mathcal{D}$ is said to be *nested** if for all U∈$\mathcal{D}$, F(U) = <F_A(U)|A∈Y(X)> is nested*.

Given an MDPFL F on an admissible set $\mathcal{D}$, an individual/criterion i∈N is said to be a (*weak*) *dictator* if for all U∈$\mathcal{D}$ and x,y∈X: U(x,i) > U(y,i) implies (x,y)∈P(F_A(U)) for all A∈Ψ(X) satisfying x,y∈A.

An MDPFL F on an admissible set $\mathcal{D}$ is said to be *weakly dictatorial* if it has a weak dictator, and *nondictatorial* otherwise (i.e., if it has no weak dictator).

Given an MDPFL F on an admissible set $\mathcal{D}$, an individual/criterion i∈N is said to have *veto power* if for all U∈$\mathcal{D}$ and x,y∈X: U(x,i) > U(y,i) implies (x,y)∈F_A(U) for all A∈Ψ(X) satisfying x,y∈A.

3. RESULTS AND DISCUSSION

To establish the main results of this paper, we will require the following implication of a marginally generalized version of a result originally due to Sen (1970). This marginally generalized version that immediately implies the subsequent results is available in Lahiri (forthcoming). The result in Sen (1970) is a generalized version of Arrow's impossibility theorem. Before we begin with the theorems, let us introduce a few concepts.

A *preference profile* (R_1,…,R_n) is a member of $\mathcal{O}^N$, where for each i∈N, R_i denotes the preferences/rankings with ties over X of individual/criterion i.

A *domain* is any nonempty subset D of $\mathcal{O}^N$.

Given a domain D an admissible set $\mathcal{D}$ is said to be D-*representable* if for all preference profiles (R_1,…, R_n)∈D, there exists an evaluation profile U∈$\mathcal{D}$ (possibly depending on (R_1,…,R_n)) such that for all i∈N, U(.,i) is a numerical representation of R_i.

An admissible set $\mathcal{D}$ is said to be *rich* if for each evaluation profile U∈$\mathcal{D}$, $\mathcal{D}$ contains every affine transformation of U.

Theorem 1: Let F be a MDPFL on an admissible set $\mathcal{D}$ where $\mathcal{D}$ is rich and $\mathcal{O}^N$-representable. Suppose F is cardinally noncomparable, globally weakly Paretian and globally binarily independent. Then, there exists i∈N, such that for all U∈$\mathcal{D}$ and x,y∈X: U(x,i) > U(y,i) implies (x,y)∈P(F_X(U)).

The only difference between the premise of Theorem 1 as stated in Lahiri (forthcoming) and the premise of the corresponding result available in Sen (1970) is that instead of assuming the admissible set to be $\mathcal{O}^N$-representable and rich as we do here, in Sen (1970) it is assumed that $\mathcal{D} = \mathcal{U}$. The two proofs under the two different assumptions are (almost) identical.

The above result is used to prove the following two results.

Theorem 2: Let F be a MDPFL on an admissible set $\mathcal{D}$ where $\mathcal{D}$ is rich and $\mathcal{O}^N$-representable. Suppose F is cardinally noncomparable, globally weakly Paretian, globally binarily independent, and nested. Then, F must be weakly dictatorial.

Proof: By Theorem 1, there exists $i \in N$, such that for all $U \in \mathcal{D}$ and $x,y \in X$: $U(x,i) > U(y,i)$ implies $(x,y) \in P(F_X(U))$. Thus, $(y,x) \notin F_X(U)$.

Suppose $A \in \Psi(X)$, with $x,y \in A$ and toward a contradiction suppose $(x,y) \notin P(F_A(U))$. As $F_A(U)$ is reflexive and complete, it must be the case that $(y,x) \in F_A(U)$. As F is nested, $F_A(U) \in F_X(U)$ and so $(y,x) \in F_X(U)$, leading to a contradiction. Thus, $(x,y) \in P(F_A(U))$ for all $A \in \Psi(X)$ satisfying $x,y \in A$ and so i is a weak dictator. Thus, F is weakly dictatorial. Q.E.D.

Corollary of Theorem 2: Let F be a MDPFL on an admissible set $\mathcal{D}$ where $\mathcal{D}$ is rich and $\mathcal{O}^N$-representable. Suppose F is cardinally noncomparable, weakly Paretian, binarily independent, and nested. Then, F must be weakly dictatorial.

Theorem 3: Let F be a MDPFL on an admissible set $\mathcal{D}$ where $\mathcal{D}$ is rich and $\mathcal{O}^N$-representable. Suppose F is cardinally noncomparable, globally weakly Paretian, globally binarily independent, and nested*. Then, there exists $i \in N$ who/that has veto power.

Proof: By Theorem 1, there exists $i \in N$, such that for all $U \in \mathcal{D}$ and $x,y \in X$: $U(x,i) > U(y,i)$ implies $(x,y) \in P(F_X(U))$. Thus, $(x,y) \in F_X(U)$.

Suppose $A \in Y(X)$, with $x,y \in A$. As F is nested*, $F_X(U) \cap (A \times A) \subset F_A(U)$. As $(x,y) \in F_X(U) \cap (A \times A)$, it must be the case that $(x,y) \in F_A(U)$. Thus, $(x,y) \in F_A(U)$ for all $A \in \Psi(X)$ satisfying $x,y \in A$ and so i has veto power. Q.E.D.

Corollary of Theorem 3: Let F be a MDPFL on an admissible set $\mathcal{D}$ where $\mathcal{D}$ is rich and $\mathcal{O}^N$-representable. Suppose F is cardinally noncomparable, weakly Paretian, binarily independent, and nested*. Then, there exists $i \in N$ who/that has veto power.

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