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Abstract

In this paper, the arbitrage opportunities in a foreign exchange market are detected using analytic hierarchy process and linear matrix inequality methods. For this purpose, first, criteria are proposed to detect the direct, triangular, quadrangular, and other types of arbitrage suspect existing in a foreign exchange market. Subsequently, the optimal arbitrage paths are given. Some simulated examples are given. A real data set is analyzed as well. Finally, a conclusion section is given.

Keywords: AHP; Arbitrage; Foreign exchange market; LMI.

1. INTRODUCTION

Arbitrage is the opportunity of riskless profit. No arbitrage is a fundamental assumption for financial derivatives pricing. In a foreign exchange market, the difference in a specific exchange rate in two or more exchange markets causes the arbitrage opportunity. Therefore, detecting the arbitrage opportunity in foreign exchange markets is an important issue and has also received considerable attention in the literature. Moosa (2003) proposed necessary and sufficient conditions for existence of the arbitrage opportunities in several types of markets. Ma (2008) applied the analytic hierarchy process (AHP) method for detecting arbitrage opportunities by applying the concept of matrix consistency. Soon and Ye (2011) detected the currency arbitrage opportunities by applying the binary integer programming method. Zhang (2012) considered an artificial neural network approach for this purpose. Habibi (2016) found the optimal arbitrage path in a given market using Markov chain approach. Moreover, he proposed the game theoretic solutions.

In the current paper, two methods, namely, AHP and linear matrix inequality (LMI), are used for arbitrage detection. Previous results of AHP are extended, and it is shown that the LMI is also a useful tool for this purpose. Generally, the AHP method uses the pair-wise comparison matrix $A = A_{n \times n} = (a_{ij})_{i,j=1,\dots,n}$ for weighting n criteria for choosing the best option. The results of AHP are valid, if the comparison matrix A is consistent. The matrix A is consistent if and only if, for every $i, j, k = 1, \dots, n$, then $a_{ij} = a_{ik}a_{kj}$. It is easy to see that for a consistent matrix A , we have

$$A^2 = nA, A^3 = n^2A, \text{ and } A^j = n^{j-1}A, j \geq 2.$$

LMI is described completely, in Section 2. The remainder of the paper is designed as follows: In the next section, first, the arbitrage detection is surveyed using an AHP-based method. Then, an LMI-based approach is proposed for arbitrage detection. Finally, the optimal arbitrage paths for several types of arbitrage cases are proposed. Section 3 presents some simulated examples. A real data set is considered as well. Conclusions are given in Section 4.

2. THEORETICAL RESULTS

Here, the arbitrage opportunities are detected in a given foreign exchange market using two methods.

2.1. AHP-Based Approach

Given foreign exchange market containing n currencies, let a_{ij} be the exchange rate of the i -th currency with respect to the j -th currency. Indeed, in the case of foreign exchange market, the pair-wise comparison matrix A becomes the exchange rate matrix where its elements as well as their eigenvalues are random. Then, the direct opportunity does not exist if $a_{ij} = a_{ik}a_{kj}$ for all $k = 1, \dots, n$. Thus, the results for a consistent matrix are translated straightforward to the arbitrage opportunities. Ma (2008) used the maximum eigenvalue of Saaty (1980) criteria to detect the arbitrage opportunities. The main idea that exists in the heart of the current AHP-based method is that it uses the Taylor expansion of a consistent matrix.

Proposition 1. If matrix A is a consistent matrix, then

(a) The exponential power of A (i.e., e^A) is given by

$$e^A = I + A + \frac{nA}{2!} + \frac{n^2A}{3!} = I + \frac{e^n - 1}{n} A,$$

where I is a $n \times n$ identity matrix.

(b) Let (λ_j, γ_j) be the j -th eigenvalue and eigenvector of matrix A . Then,

$$e^{\lambda_j} = 1 + \frac{e^n - 1}{n} \lambda_j.$$

Proof. Using identity $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$, notice that the proof of part (a) is straightforward. For part (b), notice that according to the real Schur decomposition (see Meyer, 2000),

$$A = \sum_{j=1}^n \lambda_j \gamma_j \gamma_j',$$

$$e^A = \sum_{j=1}^n e^{\lambda_j} \gamma_j \gamma_j'.$$

Substituting these relations in the equation $e^A - \frac{e^n - 1}{n} A = I$ and using the orthogonal properties of γ_j 's, the proof is clear.

A consistent matrix A has the largest eigenvalue equal to n , and the other eigenvalues are zero. One can see that both 0 and n are roots of equation $e^{\lambda_j} - 1 - \frac{e^n - 1}{n} \lambda_j$. To study some features of the current equation, consider the sequence of real-valued functions $f_n(x)$ as follows:

$$f_n(x) = e^x - \frac{e^n - 1}{n} x - 1.$$

Some features of $f_n(x)$ are as follows:

- (i) Notice that $f_n(0) = f_n(n) = 0$. Then, Rolle's theorem (see Browder, 2001) over the interval $(0, n)$ implies that there exists a $x^* \in (0, n)$ such that $e^{x^*} = \frac{e^n - 1}{n}$.
- (ii) One can see that $f_n(x)$ is approximated by

$$\frac{x^2}{2} + \left(1 - \frac{e^n - 1}{n}\right)x,$$

which has two roots: zero and $e^n - 3 \geq 0$ for $n \geq 2$.

- (iii) For x 's between two roots, the $f_n(x)$ is negative, and it is positive otherwise. As $e^n - 3 \geq n$, $f_n(x)$ is negative for $x \in (0, n)$ and is positive for x 's outside of this interval. It is also seen that $\int_0^n f_n(x) dx = -n$.

Proposition 2. Functions $f_n(x)$, $n > 1$, are approximated as follows:

$$f_n(x) = x(x - n) \left\{ \frac{1}{2} + \frac{1}{3!} \frac{x^2 - n^2}{x - n} + \frac{1}{4!} \frac{x^3 - n^3}{x - n} + \dots \right\}.$$

Proof. It is only enough to use the Taylor expansions of e^x and $\frac{e^n - 1}{n}$.

Remark. When $x \in (0, n)$, $f_n(x)$ is negative and is positive for x 's outside of this interval. Figures 1a and 1b are plots of $f_3(x)$ and $f_4(x)$, respectively, which show these facts.

The probability of existence of arbitrage is given as follows. For this purpose, notice that the equation $A^2 = nA$ is equivalent to the no arbitrage assumption. In addition, notice that the nonzero elements $A - \frac{A^2}{n}$ characterize the direct arbitrage opportunity. Moreover, nonzero elements of $A - \frac{A^3}{n^2}$ distinguish the triangular arbitrage opportunities. Equivalently, nonzero coefficients of

$$e^{\lambda_j} - \frac{e^n - 1}{n} \lambda_j - 1,$$

include the direct and triangular arbitrage opportunities. Next, notice that λ_j 's are random variables. Denote their means and variances by μ_j and σ_j^2 , $j = 1, \dots, n$. The following points are valuable.

Figure 1a. Plot of $f_3(x)$.

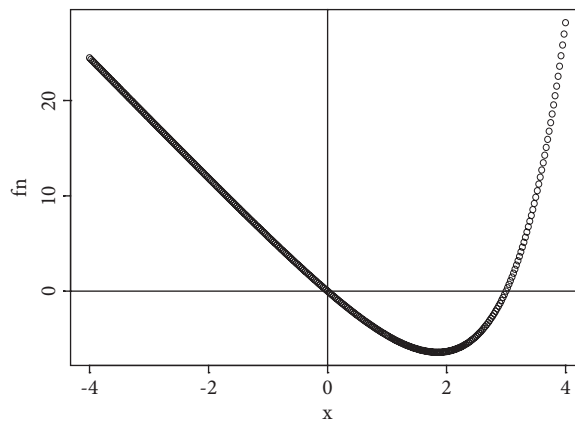
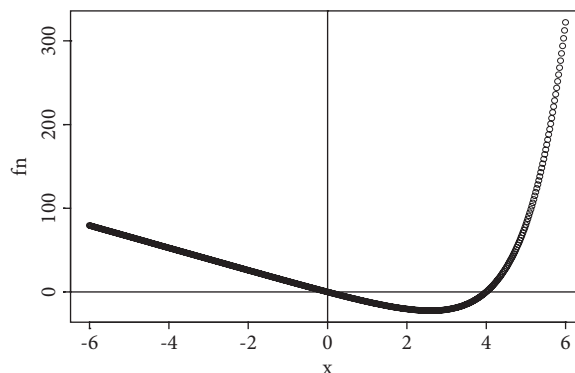


Figure 1b. Plot of $f_4(x)$.



(i) Using the delta method, the means and variances of $f_n(\lambda_j)$ are approximated as follows:

$$\begin{cases} E(f_n(\lambda_j)) \approx f_n(\mu_j) + \frac{\sigma_j^2}{2} f_n''(\mu_j), \\ \text{var}(f_n(\lambda_j)) = \sigma_j^2 (f_n'(\mu_j))^2, \end{cases}$$

where $f_n'(x) = e^x - \frac{e^n - 1}{n}$, $f_n''(x) = e^x$.

(ii) Using the Chebyshev's inequality, the probability that $f_n(\lambda_j)$ being nonzero is upper bounded as follows:

$$P(|f_n(\lambda_j)| > \varepsilon) \leq \frac{\text{var}(f_n(\lambda_j)) + E^2(f_n(\lambda_j))}{\varepsilon^2}.$$

(iii) Triangular arbitrage exists if the third coefficient of Taylor expansion of $f_n(\lambda_j)$ is nonzero for some λ_j 's. Other types of arbitrage (say, triangular, quadrangular, etc.) are detected, similarly.

A practical rule is given in the following for detecting several types of arbitrage opportunities.

Practical rule. To detect arbitrage opportunity in a foreign exchange market, first, calculate $P(|f_n(\lambda_j)| > \varepsilon)$. If this value is negligible, then there is no arbitrage. If this probability is far from zero, then check its coefficient. Nonzero coefficients indicate the corresponding type of arbitrage.

2.2. LMI-Based Approach

Here, an LMI-based approach is proposed for detecting arbitrage opportunity (see Isermann, 2006). In the previous section, it was seen that no type of arbitrage does exist if $e^A - I - \frac{e^n - 1}{n}A$ is zero. One can see that

$$e^A - I - \frac{e^n - 1}{n}A = \sum_{j=1}^n f_n(\lambda_j) \gamma_j \gamma_j'.$$

Under the no arbitrage assumption, $f_n(\lambda_j)$ is zero. When there is a type of arbitrage, $f_n(\lambda_j) \neq 0$ (which is positive or negative, $f_n(\lambda_j) \gtrless 0$). That is, under the no arbitrage assumption, $f_n(\lambda_j)$ is far from zero. Then,

$$\left(1 - \frac{e^n - 1}{n}\right)A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots \gtrless 0.$$

Existence of any type of arbitrage is equivalent to find the coefficients x_1, x_2, \dots , such that

$$x_1 A + x_2 A^2 + \dots \gtrless 0.$$

It is easy to see that if $x_1 \gtrless \frac{e^n - 1}{n} - 1$, $x_j \gtrless \frac{1}{j!}$, $j \geq n$, then there is an arbitrage opportunity.

Proposition 3. For a given exchange rate matrix A , there is an arbitrage opportunity if coefficients x_1, x_2, \dots exist, such that

$$x_1 A + x_2 A^2 + \dots \gtrless 0,$$

at which $x_1 \gtrless \frac{e^n - 1}{n} - 1$, $x_j \gtrless \frac{1}{j!}$, $j \geq 2$.

The above results show that the existence of any type of arbitrage can be formulated as an LMI problem. Hereafter, the solution of the above LMI problem is investigated. For this purpose, the Schur decomposition is applied. Therefore, the above LMI problem becomes

$$\sum_{j=1}^n (x_1 \lambda_j + x_2 \lambda_j^2 + \dots) \gamma_j \gamma_j' \gtrless 0.$$

It is equivalent to find x_j 's, such that $(x_1\lambda_j + x_2\lambda_j^2 + \dots) \geq 0$.

Again, as shown in the following, a practical rule is proposed for arbitrage detection using the LMI formulation.

Practical rule. A simple solution is to let $x_j = \frac{1}{j!}, j \geq 2$. Then, $x_1\lambda_j + e^{\lambda_j} - \lambda_j - 1 \geq 0$. As soon as x_1 is found, if $x_1 > \frac{e^n - 1}{n} - 1$, then there is an arbitrage opportunity but if $x_1 = \frac{e^n - 1}{n} - 1$, then there is no arbitrage in the foreign exchange market.

2.3. Optimal Arbitrage Path

Here, the optimal arbitrage path is studied. Notice that the direct arbitrage gain is obtained via optimization problem defined by

$$\max_{\{w_k\}_{k=1}^n} \sum_{k=1}^n w_k (a_{ij} - a_{ik} a_{kj}),$$

vs $\sum_{k=1}^n w_k = 1$. Habibi (2016) showed that $w_{k^*} = 1$ for k^* , such that

$$a_{ij} - a_{ik^*} a_{k^*j} = \max_k (a_{ij} - a_{ik} a_{kj}).$$

Considering $w_k = 1/n$, the arbitrage gain is

$$a_{ij} - \frac{1}{n} \sum_{k=1}^n a_{ik} a_{kj} = 0.$$

For a quadrangular arbitrage case, it is enough to find $k_i, i = 1, 2, 3$, such that

$$\max_{\{k_1\}, \{k_2\}} (a_{ij} - a_{ik_1} a_{k_1k_2} a_{k_2k_3} a_{k_3j})$$

is achieved. Again, changing the problem to the iterated optimization problem, the winning strategy is derived. A triangular arbitrage is detected via double optimization problem as follows:

$$\max_{\{k_1\}, \{k_2\}} (a_{ij} - a_{ik_1} a_{k_1k_2} a_{k_2j}).$$

It is equivalent to the following iterated optimization:

$$\begin{aligned} &\max_{\{k_1\}} \max_{\{k_2\}} (a_{ij} - a_{ik_1} a_{k_1k_2} a_{k_2j}) = \\ &\max_{\{k_1\}} a_{ik_1} \max_{\{k_2\}} (a_{k_1j} - a_{k_1k_2} a_{k_2j}) = \\ &\max_{\{k_1\}} a_{ik_1} \max_{\{k_2\}} (a_{k_1j} - a_{k_1k_2^*} a_{k_2^*j}) = \\ &\max_{\{k_1\}} (a_{ij} - a_{ik_1} a_{k_1k_2^*} a_{k_2^*j}) = \\ &a_{k_2^*j} \max_{\{k_1\}} (a_{ik_2^*} - a_{ik_1} a_{k_1k_2^*}) = \\ &a_{ij} - a_{ik_1^*} a_{k_1^*k_2^*} a_{k_2^*j}. \end{aligned}$$

3. APPLIED RESULTS

In this section, the above discussions are implemented in practice in two ways: First, they are applied in the simulated foreign exchange market. Then, a real data set is considered.

3.1. Simulations

Consider the foreign exchange market containing three currencies with intrinsic value $g_1 = 2, g_2 = 4$, and $g_3 = 8$. Following Ma (2008), $a_{ij} = \frac{g_i}{g_j}, i, j = 1, 2, 3$. Then, the exchange rate matrix A is given as follows:

$$A = \begin{pmatrix} 1 & 0.5 & 0.25 \\ 2 & 1 & 0.5 \\ 4 & 2 & 1 \end{pmatrix}.$$

One can see that, for example, $A - \frac{A^3}{9}$ is a zero matrix, and the eigenvalues are 3,0,0 and $f_n(\lambda_j)$ is totally zero. Next, assume that A is given as A^* as follows:

$$A^* = \begin{pmatrix} 1 & 0.45 & 0.25 \\ 2 & 1 & 0.5 \\ 4 & 2 & 1 \end{pmatrix}.$$

Then, $A^{*2} - 3A^*$ is $\begin{pmatrix} -0.1 & 0.05 & -0.25 \\ 0 & -0.1 & 0 \\ 0 & -0.2 & 0 \end{pmatrix}$. The maximum component of this matrix is 0.05, which

indicates that the arbitrage path starts from a_{12} . The $f_n(\lambda_j)$ values are -0.45 , -0.18 , and zero. Next, let g_j 's are $g_1=2$, $g_2=4$, and $g_3=8$. However, $a_{ij} = \frac{g_i}{g_j} \varepsilon_{ij}$, where ε_{ij} are independent random variables with common lognormal distributions with parameters $\mu=0$, $\sigma=0.1$. The histogram (density) of maximum of eigenvalue of A (i.e., λ_{max}) is shown in Figure 2.

The probability of $p = P(|f_n(\lambda_{max})| > a)$ is given for various values of a (see Table 1).

In this case, there are many arbitrage opportunities in the foreign exchange market. The histogram (density) of $f_n(\lambda_{max})$ is shown in Figure 3.

Again, consider the exchange rate matrix A . The real Schur decomposition implies that

$$e^A = e^3 \gamma_1 \gamma_1' + \gamma_2 \gamma_2' + \gamma_3 \gamma_3'.$$

Figure 2. Density Plot of λ_{max} for $\mu = 0$, $\sigma = 0.1$.

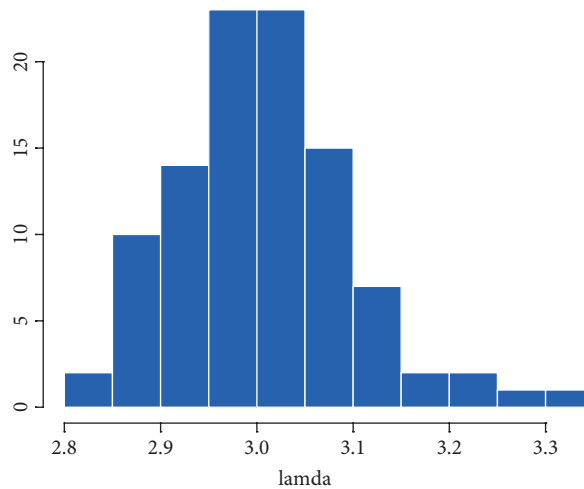


Table 1. Values of p for Various Quantiles a .

a	0.1	0.2	0.5	1	1.5	2	3.5
p	0.93	0.85	0.64	0.38	0.21	0.1	0.04

Figure 3. Density Plot of $f_n(\lambda_{max})$ for $\mu = 0, \sigma = 0.1$.

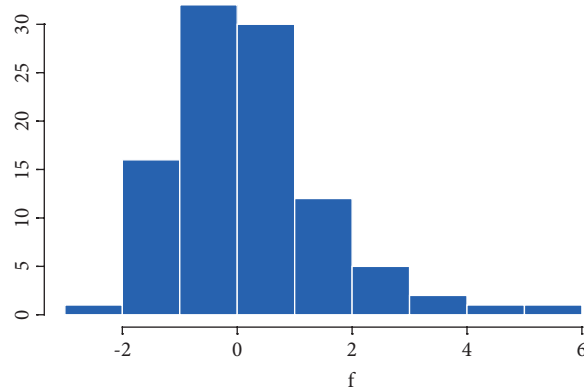
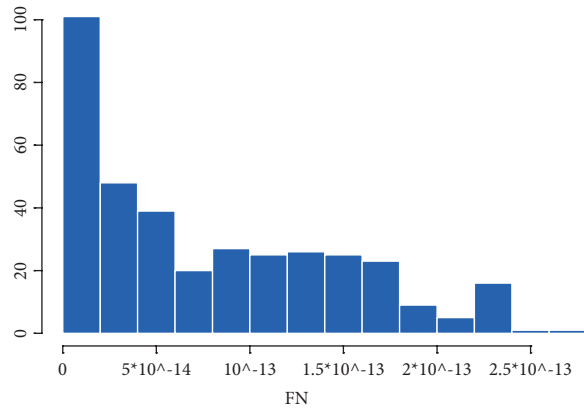


Figure 4. Histogram of $\max_{1 \leq j \leq n} |f_n(\lambda_j)|$.



Then, one can see that

$$\left(1 - \frac{e^3 - 1}{3}\right)A + \frac{A^2}{2!} + \dots + \frac{A^6}{6!} \approx 0.$$

This fact indicates that there is no arbitrage opportunity in the given market. However, this is not true for the exchange rate matrix A^* . This indicates that there is at least one arbitrage opportunity in the market.

3.2. Real Data

In previous sections, it was seen that there are some arbitrage opportunities if for some λ_j , then $|f_n(\lambda_j)| \neq 0$. The probability of arbitrage is given by

$$p = Pr(\max_{1 \leq j \leq n} |f_n(\lambda_j)| > a),$$

where a is a small number. Here, this probability is computed for a foreign exchange market. The market contains three currencies including USD, GBP, and EUR with intrinsic values $g_1, g_2,$ and $g_3,$ respectively, and $a_{ij} = \frac{g_i}{g_j}$. The data are daily exchange rates for period study of 16/06/2016 to 16/06/2017. The histogram (density) of variable $\max_{1 \leq j \leq n} |f_n(\lambda_j)|$ is shown in Figure 4.

Table 2. Values of p for Various Quantiles a .

a	10^{-5}	10^{-10}	10^{-13}
p	0	0	0.357

It is observed that most values of $\max_{1 \leq j \leq n} |f_n(\lambda_j)|$ are zero indicating that there is no arbitrage opportunity. Table 2 provides the values of p for various values of a .

In this case, there is no arbitrage opportunity in the foreign exchange market.

4. CONCLUSIONS

This paper studies the existence of arbitrage opportunity in a given foreign exchange market. Two fundamental approaches including the AHP and LMI are applied to detect the arbitrages. However, both the approaches reduce to the $\max_{1 \leq j \leq n} |f_n(\lambda_j)|$ as a suitable tool for distinguishing the arbitrage opportunities where $f_n(x) = e^x - \frac{e^n - 1}{n}x - 1$. This criterion is an alternative instrument for the maximum eigenvalue of exchange rate matrix of Saaty (1980) applied by Ma (2008). The simulation results reveal that the method works well.

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